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Decision-making under uncertainty – a quantum value operator approach

Lizhi Xin[1] · **Houwen Xin**[2]

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Abstract
Decision-making under uncertainty is the unification of people's subjective beliefs and the objective world. Quantum value operator is proposed to simulate people's decision process. Quantum value operator guides people to choose corresponding actions based on their subjective beliefs through objective world. The quantum value operator can be constructed from basic quantum gates and logic operations as a quantum decision tree and the genetic programming is applied to optimize quantum decision trees. Quantum expected value is used as fitness function to evaluate the observed outcomes (gain or loss) in the process of decision-making under incomplete information. Basically based on Darwin's natural selection, a computational model that incorporating insights from quantum theory is proposed to describe and explain people's decision-making in the real world.

1 Introduction

In general, there are multiple actions available for decision-making. Decision-making can be viewed as a two-phase process: evaluation process and selection process. People predict the consequences of each action, evaluate which action will maximize value, and then make a selection. Classical decision theory holds that rational-economic person knows the utility function as well as probability distribution through evaluation process then maximizes utility by an optimal selection [1–3]. Information completeness and selection consistence are required for expected utility theory, however in the real world information is rarely complete and consistency of selection cannot be guaranteed due to complexity and people show irrational behaviors [4–7] which cannot be explained by classical decision theory.

Classical decision theory is a "black box"; people do not know what really happens inside the box. Scientists are trying to apply quantum theory to reveal how decisions are made. Recently many quantum-like decision theories [8–10] have been proposed based on quantum probability instead of classical probability to revise the mathematical structure that's used in classical models. Aerts et al. first proposed to apply quantum

✉ Houwen Xin
hxin@ustc.edu.cn

[1] Hefei, People's Republic of China

[2] Department of Chemical Physics USTC, Hefei, Anhui, People's Republic of China

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probability in decision theories [11, 12]; Busemeyer et al. proposed a quantum-like model to describe human judgments and the order effect [13–15]; Khrennikov et al. improved the Busemeyer quantum model by applying quantum instrument of quantum measurement theory [16–20]; Yukalov et al. proposed a rigorously axiomatic quantum decision theory [21–23]. Almost all mentioned well-developed quantum-like decision models applied the rigorous mathematical structure of quantum theory and discussed about the presence of the quantum-like interference effects in the process of decision-making under uncertainty.

We are firm believer that people's subjective belief cannot be computed by rigorous mathematical formula; instead people's belief can be learned from the past experience (historical data); the past experience is the unity of people's subjective beliefs and objective facts observed, and the observed value (gain or loss) is the bridge between those two different worlds. The decision's "black box" can be opened through the bridge of the observed value. Unlike other quantum decision-making theories that describe and explain decision-making process through rigorous mathematical structures of quantum theory (quantum probability theory), our proposed quantum decision theory discovers "laws" of thought by learning observed historical data (machine-learning by genetic programming), there is no wave function, no Schrödinger differential equation, and no quantum transition probability in our decision theory.. In this paper, quantum value operator is proposed to replace utility function to evaluate the value (gain or loss) in the process of decision-making under incomplete information and Darwin's theory of evolution is applied to construct a quantum decision tree based on natural selection (maximizing quantum expected value) to simulate the decision-making process. Quantum decision trees can be constructed from eight basic quantum gates as well as three logic operations, and it can be interpreted as a set of mixed strategies to guide people to make better decisions in the real world.

2 Quantum expected value

Usually people subjectively choose an action $a_i \in \{a_1, \cdots, a_m\}$ where nature's objective state is in $q_j \in \{q_1, \cdots, q_n\}$ when decisions were made, and the observed value v_{ij} depends on both the state of the nature and choice of brain shown in Table 1. Natural state describes the objective world; we hypothesize that an uncertain natural state can be represented by superposition of all possible states in terms of the Hilbert state space [24, 25] as in (1). Mind state describes the subjective world; we also hypothesize that

Table 1 State-action-value decision table

State Action	q_1	$\cdots$	q_j	$\cdots$	q_n
a_1	v_{11}		$\cdots$		v_{1n}
$\vdots$		$\ddots$			
a_i	$\vdots$		v_{ij}		$\vdots$
$\vdots$				$\ddots$	
a_m	v_{m1}		$\cdots$		v_{mn}

undecided mind state can be represented by superposition of all possible actions as in (2). Usually the information of decision-making under uncertainty is incomplete, the observed value can be represented by a mixed state's density operator as a quantum value operator in (3). Quantum expected value can be represented as in (4).

$$|\psi\rangle = \sum_j c_j |q_j\rangle, \sum_j |c_j|^2 = 1 \tag{1}$$

$$|\phi\rangle = \sum_i \mu_i |a_i\rangle, \sum_i |\mu_i|^2 = 1 \tag{2}$$

$$\hat{V} = \sum_i p_i |a_i\rangle\langle a_i|, \sum_i p_i = 1 \tag{3}$$

$$< \hat{V} >= \langle\psi|\hat{V}|\psi\rangle = \sum_i p_i \sum_j |c_j|^2 |\langle a_i|q_j\rangle|^2 = \sum_i p_i \sum_j \omega_j v_{ij} \tag{4}$$

where $p_i = |\mu_i|^2$ is a person's subjective probability in choosing an action a_i, subjective probability represents the people's degree of belief in a single event; $\omega_j = |c_j|^2$ is the objective frequency at which natural state is in q_j, objective frequency represents the statistical results of multiple occurrences of objective states; value matrix $v_{ij} = |\langle a_i|q_j\rangle|^2$ is the observed value when the decision was made which a person chooses an action a_i where nature's state is in q_j. Quantum expected value (4) suggests that a subjective and objective unified expected value is obtained through people's beliefs which are based on natural states. The different actions people took leads to different expected value; in other words; the actual outcome is created based on people's subjective beliefs and objective natural states.

Before a decision-maker makes a decision, his/her mind state is in a pure state, a state in which they can decide which actions to take at the same time. This pure state is when the mind states of all actions are superposed in the decision-maker's mind. When the people make the final decision, their mind state is then transformed from that pure state into a mixed state, which is when they decide to take one and only action a_i with certain degrees of belief p_i. Basically, this transformation is the brain choosing from one of the available actions as in (5).

$$\text{Decision}: |\phi\rangle\langle\phi| \rightarrow \hat{V} = \sum_i p_i |a_i\rangle\langle a_i| \tag{5}$$

As an example, we can represent the state of future market in terms of the Hilbert state space as in (6), trader's mind state is represented as in (7), and quantum value operator which projects a trader's beliefs onto an action of buying or selling a security is represented as in (8).

$$|\psi\rangle = c_1|q_1\rangle + c_2|q_2\rangle \tag{6}$$

$$|\phi\rangle = \mu_1|a_1\rangle + \mu_2|a_2\rangle \tag{7}$$

$$\hat{V} = p_1|a_1\rangle\langle a_1| + p_2|a_2\rangle\langle a_2| \tag{8}$$

where $|q_1\rangle$ indicates a state in which the market is up and $|q_2\rangle$ indicates a state in which the market is down;$|a_1\rangle$ represents trader's action to buy and $|a_2\rangle$ represents trader's action to sell;p_1 represents the subjective probability which a trader choose to buy and p_2 represents the subjective probability which a trader choose to sell.

The process of trader's decision-making is as in (9):

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \xrightarrow{\text{diagonalization}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \xrightarrow{\text{normalization}} \hat{V} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} = p_1|a_1\rangle\langle a_1| + p_2|a_2\rangle\langle a_2| \tag{9a}$$

$$|a_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |a_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; |a_1\rangle\langle a_1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, |a_2\rangle\langle a_2| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{9b}$$

3 Quantum decision tree (qDT)

A quantum value operator can be constructed from the basic quantum gates [26] and logic operations to form a qDT. The qDT composes of different nodes and branches. There are two types of nodes, non-leaf nodes and leaf nodes. The non-leaf nodes are composed of the operation set F as in (10); the leaf nodes are composed of the data set T as in (11) and (12). The construction process of a qDT is to randomly select a logic symbol from the operation set F as the root of the qDT, and then grows corresponding branches according to the nature of the operation symbol and so on until a leaf node is reached.

$$F = \{+(\text{ADD}), *\,(\text{MULTIPLY}), //(\text{OR})\} \tag{10}$$

$$T = \{H, X, Y, Z, S, D, T, I\} \tag{11}$$

$$\left\{ \begin{array}{l} H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \right\} \tag{12}$$

4 Quantum genetic programming (qGP)

A tree structure is used for encoding by genetic programming (GP) [27, 28], which is particularly appropriate to solve hierarchical and structured complex problems. The qDT can be optimized by the qGP as shown in Fig. 1. If market is up and a trader chooses to buy, the value V_k is positive and the trader would make a profit, if the trader choose to sell, the value V_k is negative and the trader would lose money; if market is down and a trader chooses to sell, the value V_k is positive and the trader would make a profit, if the trader choose to buy, the value V_k is negative and the trader would lose money. The purpose of QGP iterative evolution is to find a satisfactory qDT through learning historical data.

4.1 qGP algorithm

Input:

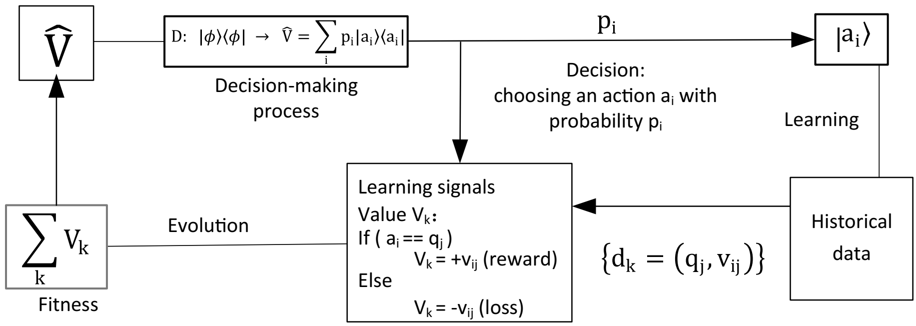

Fig. 1 The evolution of the quantum value operator

- Historical data set $\{d_k = (q_k, v_k)\}$ which includes N samples, each sample consists of natural state and value.
- Setting

  1) Operation set $F = \{+, *, //\}$
  2) Data set $T = \{H, X, Y, Z, S, D, T, I\}$, eight basic quantum gates
  3) Crossover probability = 70%; Mutation probability = 5%.

Initialization:

- Population: randomly create 300 qDTs.

Evolution:

- for $i = 0$ to n ($n = 100$ generations):

  a) Calculate fitness for each qDT based on historical data set.
  b) According to the quality of fitness:

     i Selection: selecting parent qDTs.
     ii Crossover: generate a new offspring using the roulette algorithm based on crossover probability.
     iii Mutation: randomly modify parent qDT based on mutation probability.

Output:

- A qDT of the best fitness.

An optimization problem mainly includes the selection of evaluation function and the acquisition of optimal solution. The evaluation function of qDT is a fitness function (15) based on quantum expected value V_k (13), and the optimal solution is obtained through continuous evolution by using selection, crossover and mutation as in (16) and qGP algorithm; $\max_{\text{reward}}\{V_i\}$ is the largest continuous rewards, $\max_{\text{loss}}\{V_j\}$ is the largest continuous loss. β is the winning rate.

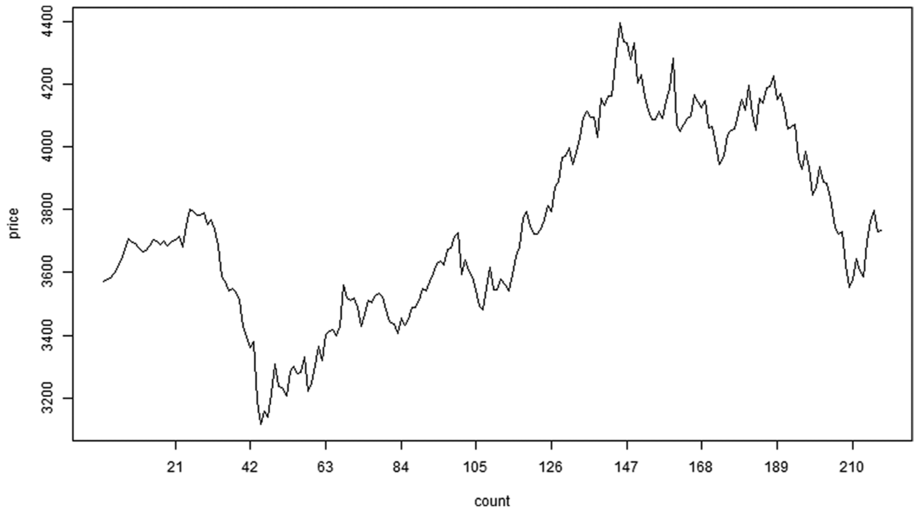

Fig. 2 Price fluctuation of rebar (contract rb1901) traded on Shanghai Futures Exchange

$$< V_k >= p_i \omega_j \langle q_j | A_i | q_j \rangle, A_i = |a_i\rangle\langle a_i| \tag{13}$$

$$\langle q_j | A_i | q_j \rangle = \langle q_j || a_i \rangle \langle a_i || q_j \rangle = |\langle a_i | q_j \rangle|^2 = v_{ij} = \begin{cases} -v, i \neq j \\ v, i = j \end{cases} \tag{14}$$

$$f_{fitness} = \beta \left(\sum_{k=0}^{N} < V_k > \right) \left(\frac{\max_{reward} \{V_k\}}{\max_{loss} \{V_k\}} \right) \tag{15}$$

$$qDT \xrightarrow{evolution} \underset{qDT \in (F \cup T)}{\operatorname{argmax}} \left(f_{fitness} \right) \tag{16}$$

The k-data of rebar contract rb1901 from 2018/1/16 to 2018/12/7 traded on the Shanghai Futures Exchange is used as the historical data shown in Fig. 2; the subjective beliefs of the trader simulated by qDT in each transaction are shown in Fig. 3. The optimized qDT after 100 generations of evolution is shown in Fig. 4. Based on the qDT as in (17) (Fig. 4), there are eight strategies with different subjective beliefs that the trader took.

The qDT which simulates people's decision process can be interpreted as a mixed strategy. Each qDT includes a set of strategies (in this case, eight strategies), each of which is a mixed density operator ($\sum_i p_i |a_i\rangle\langle a_i|$); Each time you make a decision, first choose a strategy S_i, and then choose an action a_i (buy or sell) based on the degree of belief $p_i(p_1$ or $p_2)$. Detail information of the first eight transactions is shown in Table 2. For the first transaction strategy S_1 was applied by the trader to buy with 88% belief, the trader loss money because the state of market is down $\left(v_{ij} = -v; i \neq j\right)$; for the seventh transaction strategy S_1 was applied by the trader to buy with 88% belief, this time the trader make a profit because the state of market is up $\left(v_{ij} = v; i = j\right)$; for the eighth transaction strategy S_7 was applied

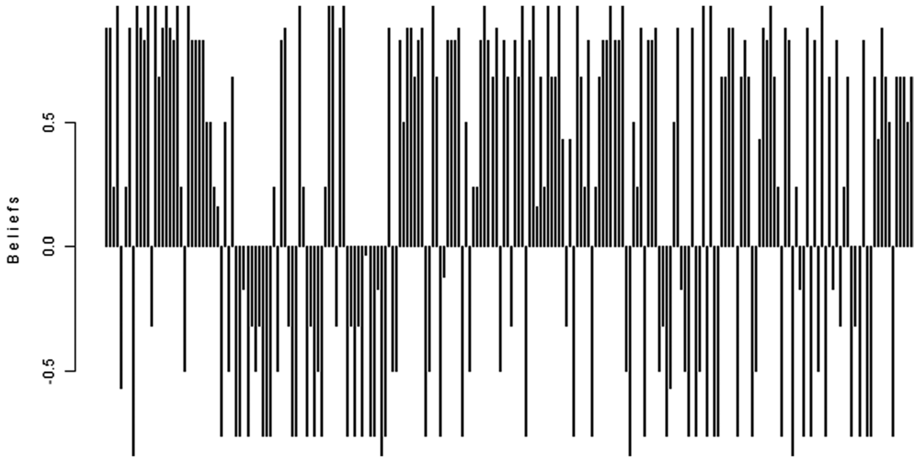

Fig. 3 A trader's beliefs of each transaction simulated by the qDT (positive: buy, negative: sell)

by the trader to sell with 84% belief, the trader make a profit because the state of market is down $\left(v_{ij} = v; i = j\right)$.

$$qDT = ((D * (Y + (((((X//(D * (S + Z)))//I)//I)//(T + H)) + T)))//(((I * Y) + (T//Y)) + (I//Y))) \tag{17}$$

- $S_1 = (((I * Y) + T) + I) \rightarrow \hat{V} = 0.88|a_1\rangle\langle a_1| + 0.12|a_2\rangle\langle a_2|$(88% belief to buy, 12% belief to sell)

- $S_2 = (((I * Y) + T) + Y) \rightarrow \hat{V} = 0.83|a_1\rangle\langle a_1| + 0.17|a_2\rangle\langle a_2|$(83% belief to buy, 17%belief to sell)

- $S_3 = (((I * Y) + H) + I) \rightarrow \hat{V} = 0.97|a_1\rangle\langle a_1| + 0.03|a_2\rangle\langle a_2|$(97% belief to buy, 3% belief to sell)

- $S_4 = (((I * Y) + H) + Y) \rightarrow \hat{V} = 0.5|a_1\rangle\langle a_1| + 0.5|a_2\rangle\langle a_2|$(50% belief to buy, 50% belief to sell)

- $S_5 = (D * (Y + (X + T))) \rightarrow \hat{V} = 0.43|a_1\rangle\langle a_1| + 0.57|a_2\rangle\langle a_2|$(43% belief to buy, 57% belief to sell)

- $S_6 = (D * (Y + (I + T))) \rightarrow \hat{V} = 0.68\langle a_1| + 0.32|a_2\rangle\langle a_2|$(68% belief to buy, 32% belief to sell)

- $S_7 = (D * (Y + ((D * (S + Z)) + T))) \rightarrow \hat{V} = 0.16|a_1\rangle\langle a_1| + 0.84|a_2\rangle\langle a_2|$(16 %belief to buy, 84 % belief to sell)

- $S_8 = (D * (Y + ((T + H) + T))) \rightarrow \hat{V} = 0.24|a_1\rangle\langle a_1| + 0.76|a_2\rangle\langle a_2|$(24% belief to buy, 76% belief to sell)

5 Discussion

Human beings record a large amount of data through the observation of the world. It is through the study of the recorded data that human beings gradually understand the objective world and build a simplified subjective "world model" in the brain. People make

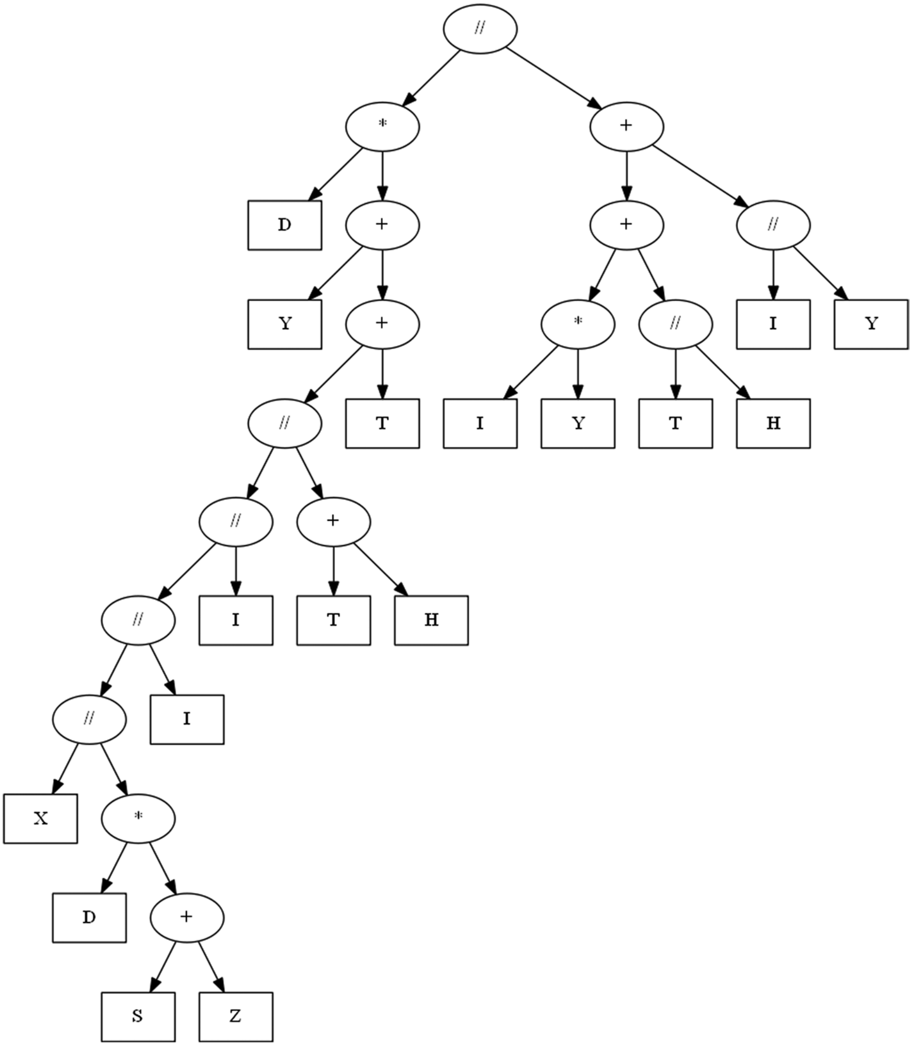

Fig. 4 An optimized quantum decision tree

Table 2 Details of the first eight transactions

Market State	Action Selected	Strategy	Expected Value	Belief
down	buy	$S_1 = (((I * Y) + T) + I)$	-8.0	88%
up	buy	$S_1 = (((I * Y) + T) + I)$	10.0	88%
up	buy	$S_8 = (D * (Y + ((T + H) + T)))$	0.0	24%
up	buy	$S_3 = (((I * Y) + H) + I)$	3.0	97%
up	sell	$S_5 = (D * (Y + (X + T)))$	-5.0	57%
up	buy	$S_8 = (D * (Y + ((T + H) + T)))$	3.0	24%
up	buy	$S_1 = (((I * Y) + T) + I)$	16.0	88%
down	sell	$S_7 = (D * (Y + ((D * (S + Z)) + T)))$	2.0	84%

decisions by considering both the world's objectivity and the subjectivity of their beliefs. Observed value is a bridge between objective world and subjective beliefs.

Von Neumann's expected utility decision theory is based on objective frequency, and Savage's theory is about subjective beliefs. Based on the state superposition of quantum theory, we propose a subjective and objective unified quantum decision theory, and a computational model not rigorous mathematical model is proposed to describe and explain decision-making under uncertainty. Instead of utility function used in classical decision theory, quantum value operator (quantum decision tree) is proposed to simulate brain's decision process under incomplete information. A quantum decision tree computes the probability of taking an action due to incomplete information and the selected actions usually cannot be given by a definite cause, but can only be obtained probabilistically through machine-learning historical data. The beliefs of decision makers are dynamically changing, and we believe that quantum value operator more realistically simulates the people's decision process in the real world.

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Authors' Contributions All authors conducted the research and contributed to the development of the model. All authors reviewed the manuscript.

Data Availability The data that support the findings of this study are available on request.

Declarations

Ethical Approval and Consent to participate Consent for publication of this paper was obtained from all authors.

Consent for Publication Consent was obtained from all authors.

Competing Interests The authors declare no competing interests.

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